Math 112: Introductory Real Analysis

Last time: • Differentiation

$$f'(x) := \lim_{t \to x} \frac{f(t)-f(x)}{t-x}$$

• Chain rule

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

• Mean value theorems

Today: l'Hospital's rule & Taylor's theorem

<u>Thm</u> (l'Hospital's rule) Suppose f and g are real, differentiable functions on (a,b), and $g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty \leq a < b \leq +\infty$.

Suppose $\lim_{x \to a} \frac{f'(x)}{g'(x)} = A$. $\quad (-\infty \leq A \leq +\infty)$

If either $\underbrace{\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0}_{①}$ or $\underbrace{\lim_{x \to a} g(x) = +\infty}_{②}$,

then $\lim_{x \to a} \frac{f(x)}{g(x)} = A$.

proof) First consider the case in which $-\infty \leq A < +\infty$.

Let q be any real number such that $A < q$.

We'll show that there exists $c \in (a,b)$ such that $\frac{f(x)}{g(x)} < q$ for all $x \in (a, c)$.

Choose r such that $A < r < q$.

Since $\lim_{x \to a} \frac{f'(x)}{g'(x)} = A < r$, there is $c_1 \in (a,b)$ such that $\frac{f'(x)}{g'(x)} < r$ for all $x \in (a, c_1)$.

2/ (proof continued)

If $a < x < y < c_1$, then by the generalized mean value theorem, there is $t \in (x,y)$ such that

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r.$$

If ① holds, then by letting $x \to a$ in the above inequality, we see that

$$\frac{f(y)}{g(y)} \leq r < q \qquad \text{for all } y \in (a, c_1).$$

If ② holds, then we can choose $c_2 \in (a, y)$ such that

$$g(x) > \max\{g(y), 0\} \qquad \text{for all } x \in (a, c_2).$$

From $f(x) - f(y) < r(g(x) - g(y))$, we obtain

$$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \qquad \text{for all } x \in (a, c_2).$$

Since the RHS $\to r$ as $x \to a$, there is $c_3 \in (a, c_2)$ such that

$$\frac{f(x)}{g(x)} < q \qquad \text{for all } x \in (a, c_3).$$

Summing up, for any q with $A < q$, there is $c \in (a,b)$ such that $\frac{f(x)}{g(x)} < q$ for all $x \in (a,c)$.

In the same manner, if $-\infty < A \leq +\infty$, and if $p < A$, then there is c' such that $p < \frac{f(x)}{g(x)}$ for all $x \in (a, c')$. It follows that $\lim_{x \to a} \frac{f(x)}{g(x)} = A$. ∎

3/

Def If f has a derivative f' on an interval, and if f' is itself differentiable, we denote the derivative of f' by f'' and call it the second derivative of f.

Continuing in this manner, we obtain functions

$$f,\ f',\ f'',\ f^{(3)},\ \cdots,\ f^{(n)},$$

each of which is the derivative of the preceding one.

$f^{(n)}$ is called the n-th derivative of f.

Thm (Taylor's theorem) Suppose f is a real function on $[a,b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a,b]$, $f^{(n)}(t)$ exists for all $t \in (a,b)$.

Let α, β be points such that $a \leq \alpha < \beta \leq b$, and define

$$P(t) := \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k .$$

(approximation of f near α by a polynomial of degree $n-1$)

Then there is $x \in (\alpha, \beta)$ such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n .$$

Rmk For $n=1$, this says $f(\beta) = f(\alpha) + f'(x)(\beta-\alpha)$ for some $x \in (\alpha,\beta)$, which is just the mean value theorem.

In general, the theorem says that f can be approximated by a polynomial of degree $n-1$, and that we can estimate the error in terms of bounds on $|f^{(n)}(x)|$.

4/ proof of Taylor's theorem)

Let $M = \dfrac{f(\beta) - P(\beta)}{(\beta - \alpha)^n}$ so that $f(\beta) = P(\beta) + M(\beta - \alpha)^n$.

We have to show that $n!\,M = f^{(n)}(x)$ for some $x \in (\alpha, \beta)$.

Put $g(t) := f(t) - P(t) - M(t-\alpha)^n \quad (a \leq t \leq b)$.

Then, $g^{(n)}(t) = f^{(n)}(t) - n!\,M \quad (a < t < b)$.

Hence the proof will be complete if we can show that $g^{(n)}(x) = 0$ for some $x \in (\alpha, \beta)$.

Note, since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, \cdots, n-1$,
we have $g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$.
We also have $g(\beta) = 0$ by our choice of M.
By the mean value theorem, $g'(x_1) = 0$ for some $x_1 \in (\alpha, \beta)$.
Again, by the mean value theorem, $g''(x_2) = 0$ for some $x_2 \in (\alpha, x_1)$.
After n steps, we arrive at the conclusion that

$$g^{(n)}(x_n) = 0 \text{ for some } x_n \in (\alpha, x_{n-1}) \subset (\alpha, \beta). \quad \blacksquare$$